A. Martin[173] noted that, in the least solution of $x^2-9817y^2=1$, x has 97 digits.

D. S. Hart[174] noted that $(r^2+s^2)y^2-1=\square$ for $y=m^2+n^2$, if

$$ms=rn\pm\sqrt{(r^2+s^2)n^2\pm s},$$

where one of r, s is odd and the other even, while n is to be found by trial.

Martin[174a] found the least solution of $x^2-9781y^2=1$.

S. Roberts[175] noted that if $t^2-Du^2=-1$ is solvable in integers, where $D=2^\mu\alpha^a\beta^b\cdots$, $\mu=0$ or 1 and α, β, $\cdots$ odd, then $t^2-D'u^2=-1$ is solvable, where $D'=2^\mu\alpha^{a+2p}\beta^{b+2q}\cdots$. Since any prime $4n+1$ is a D, any odd power of it is a D'. If $D=s^2d$, the solvability of $t^2-du^2=-1$ is a necessary, but not sufficient, condition for the solvability of $t^2-Du^2=-1$.

Roberts[176] proved that, if t, u are the least solutions of $t^2-Au^2=1$, there are values t_1, u_1, less than t, u, for which either $Mt_1^2-Nu_1^2=\pm1$, $MN=A$, or $Mt_1^2-Nu_1^2=\pm2$, $MN=A$, unless $M=1$. If the first of these equations is solvable and $M<N$, then M is the middle denominator of the period of the continued fraction for $\sqrt{A}$; but if the second holds, and not the first, $2M$ is the middle denominator.

H. Brocard[177] gave a bibliography and historical notes on Pell's equation.

K. E. Hoffmann[178] recalled that Lagrange proved that x_0, y_0 is a solution of $x^2-Ay^2=1$ if x_0/y_0 is the convergent corresponding to the first or first two periods of the continued fraction for $\sqrt{A}$. Other solutions follow from

$$x_n+y_n\sqrt{A}=(x_0+y_0\sqrt{A})^n.$$

While it is usually merely stated that x_n/y_n is a convergent to a later complete period, a direct proof is here given by use of the "closed form" of a periodic continued fraction (*ibid.*, 62, 1878, 310–6).

A. Kunerth[179] gave a "practical" method of solving

$$y^2=ax^2+bx+c. \tag{17}$$

If a rational solution is known, we may transform (17) into

$$y^2=(\alpha x+\beta)^2+(\gamma x+\delta)(\epsilon x+\zeta). \tag{18}$$

Hence every such transformation yields two values $-\delta/\gamma$ and $-\zeta/\epsilon$ of x giving rational solutions. If $x=m/n$, $y=r/n$ is a solution of (17), take $\gamma=n$, $\delta=-m$. Then $r=m\alpha+n\beta$, from which we may determine α, β. Then ϵ, ζ may be found from (18). To proceed without a known solution, subtract $(\alpha x+\beta)^2$ from (17) and employ the condition that the difference be a product of two linear functions:

$$(b-2\alpha\beta)^2-4(a-\alpha^2)(c-\beta^2)=\Delta^2. \tag{19}$$

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[173] The Analyst, Des Moines, 4, 1877, 154–5.
[174] *Ibid.*, 5, 1878, 118–9.
[174a] Math. Visitor, 1, 1878, 26–7.
[175] Proc. London Math. Soc., 9, 1877–8, 194.
[176] *Ibid.*, 10, 1878–9, 30–32.
[177] Nouv. Corresp. Math., 4, 1878, 161–9, 193–200, 228–232, 337–343.
[178] Archiv Math. Phys., 64, 1879, 1–8.
[179] Sitzungsber. Akad. Wiss. Wien (Math.), 78, II, 1878, 327–37.

Set $D=b^2-4ac$, $\beta=(K+b\alpha)/(2a)$. Then $K^2=a\Delta^2+D(\alpha^2-a)$. Hence we have to assign to Δ and α such values that the latter sum is a square.

To apply (pp. 338–346) this method to the congruence $y^2\equiv c \pmod b$, where b is a prime, we have (17) for $a=0$. Then (19) holds for $\Delta=b+2p\alpha$ if

$$\alpha=\frac{-bw(v+\beta w)}{v^2-cw^2}, \qquad \frac{v}{w}=p.$$

The first denominator may be made equal to $\pm b$ if $\pm b$ is a quadratic residue of c. Then $\alpha=\mp w_0(v_0+\beta w_0)$.

Kunerth[180] continued the same subject. Let α_1, β_1 be a solution of $r=m\alpha+n\beta$. Then $\alpha=\alpha_1-np$, $\beta=\beta_1+mp$. Substitute these in (18), with $\gamma=n$, $\delta=-m$. After several reductions, we get

$$-\epsilon x-\zeta=(np^2-2\alpha_1 p-\epsilon)x-(mp^2+2\beta_1 p+\zeta).$$

Then (17) has an integral solution if and only if p can be chosen to make the value of x for which the preceding vanishes an integer.

A. B. Evans and others[181] proved that, if p_n/q_n is the last convergent in the first period of the continued fraction for $\sqrt{A}$, and r is the largest integer $\leqq\sqrt{A}$, then $p_n=rq_n-q_{n-1}$. Hence we can derive x from y in a solution of $x^2-Ay^2=1$.

J. de Virieu[182] used the final digits to show that xy is divisible by 5 in

$$24x^2+1=y^2. \tag{20}$$

E. Lionnet[183] stated and M. Rocchetti and F. Pisani[183] proved easily that three successive sets (x_i, y_i) of solutions of (20) or $2x^2+1=3y^2$ satisfy $x_{n+1}=10x_n-x_{n-1}$, $y_{n+1}=10y_n-y_{n-1}$, with $(x_1, y_1)=(0,1)$ or $(1, 1)$, $(x_2, y_2)=(1,5)$ or $(11, 9)$, respectively. For solutions x of the second equation, $3x^2+2$ is of the form $360n+5$ and is simultaneously a sum of three consecutive squares and a sum of two consecutive squares. For $x^2+1=2y^2$, $x_n=6x_{n-1}-x_{n-2}$, $y_n=6y_{n-1}-y_{n-2}$, $(x_1, y_1)=(1, 1)$, $(x_2, y_2)=(7, 5)$.

S. Réalis[184] used $x^2-ky^2=(\alpha^2-k\beta^2)(A^2-kB^2)^2$, where

$$x=\alpha A^2-2k\beta AB+k\alpha B^2, \qquad y=-\beta A^2+2\alpha AB-k\beta B^2,$$

to derive a new solution of $x^2-ky^2=h$ from a given solution α, β and a solution of $A^2-kB^2=1$.

H. Poincaré[185] noted that, if m is odd, and a, b give the least integral solutions of $a^2-mb^2=1$ and c, d give the least odd integral solutions of $c^2-md^2=4$, then

$$\left(\frac{c+d\sqrt{m}}{2}\right)^3=a+b\sqrt{m}.$$

Several[186] proved easily that $x_{n+p}=2x_px_n-x_{n-p}$, $y_{n+p}=2x_py_n-y_{n-p}$, if x_n, y_n be the nth set of positive integral solutions of $x^2-Ny^2=1$ [$x_0=1$, $y_0=0$].

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180 Sitzungsber. Akad. Wiss. Wien (Math.), 82, II, 1880, 342–75.

181 Math. Quest. Educ. Times, 30, 1879, 49.

182 Nouv. Ann. Math., (2), 17, 1878, 476.

183 *Ibid.*, (2), 18, 1879, 479, 528; (2), 20, 1881, 425–7, 373–4. Cf. Pisani[63] of Ch. VII.

184 Nouv. Corresp. Math., 6, 1880, 306–312, 342–350.

185 Comptes Rendus Paris, 91, 1880, 846.

186 Math. Quest. Educ. Times, 34, 1880, 114.